Fourier Series
Case Study: Equation For A Parabola

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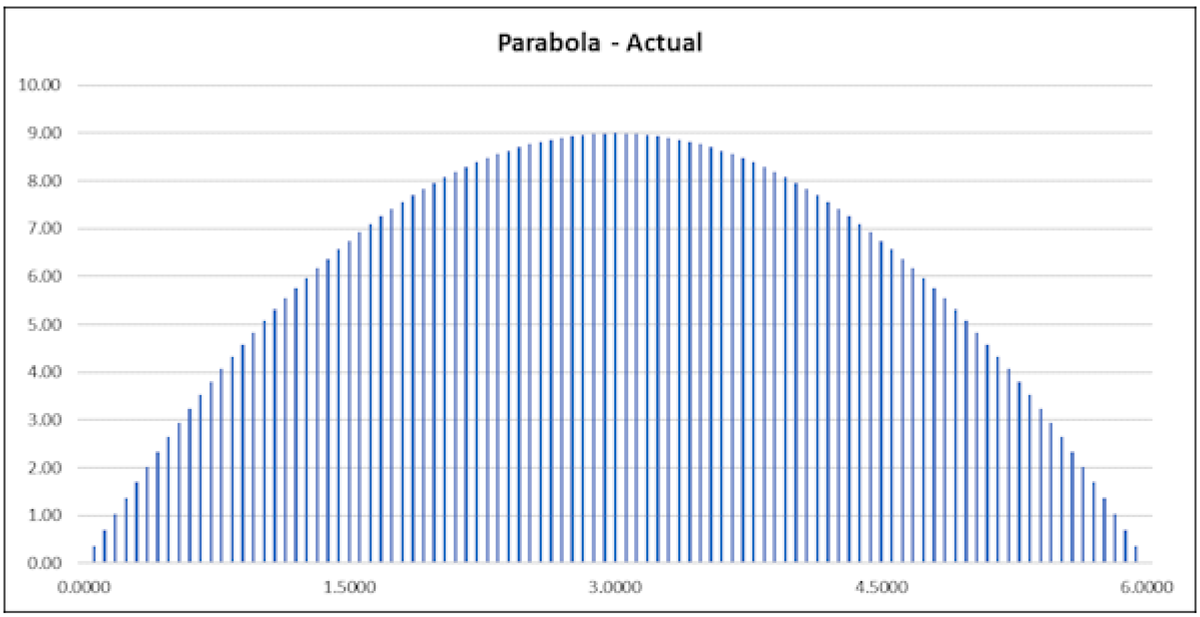
In this white paper we will use a Fourier Series to define a parabola of period 2π. To that end we will work through the following hypothetical problem...

Our Hypothetical Problem

We are tasked with building an equation for a Fourier Series that defines the following parabola...

$$f(0) = 0 \text{ ...and... } f(\pi) = 9.00 \text{ ...and... } f(2\pi) = 9.00 \text{ ...where... } 0 \le x \le 2\pi \tag{1}$$

The graph of our parabola is...



Note: The parameter values for our parbola above are...

$$a = -1.0000 \text{ ...and... } b = 6.0000 \text{ ...and... } c = 0.0000 \tag{2}$$

Questions:

1. What is our equation for the Fourier Series approximated parabola?
2. What are the values of the constant term and the first three coefficients for the sine and cosine terms?
3. Graph the Fourier Series approximated parabola after 1 iteration.
4. Graph the Fourier Series approximated parabola after 3 iterations.
5. Graph the Fourier Series approximated parabola after 100 iterations.

Building Our Model

The equation for a parabola is...

$$f(x) = a\,x^2 + b\,x + c \;\; \text{...where...} \;\; 0 \le x \le 2\,\pi \tag{3}$$

We will use the following Fourier Expansion to approximate parabola Equation (3) above... [2]

$$g(x) = A_0 + \sum_{n=1}^{\infty} A_n \cos(n\,x) + \sum_{m=1}^{\infty} B_m \sin(m\,x) \tag{4}$$

Note the solution to the following integrals... [2]

$$A_0 = \frac{1}{2\,\pi} \int_u^v f(x)\,\delta x \;\; \text{...and...} \;\; A_n = \frac{1}{\pi} \int_0^{2\,\pi} f(x)\,\cos(n\,x)\,\delta x \;\; \text{...and...} \;\; B_m = \frac{1}{\pi} \int_0^{2\,\pi} f(x)\,\sin(m\,x)\,\delta x \tag{5}$$

Using Equation (5) above and Appendix Equation (10) below, the solution to our parabola's first parameter (A_0) in Equation (4) above is...

$$A_0 = \frac{1}{2\,\pi} \int_u^v f(x)\,\delta x = \frac{1}{2\,\pi} \left(\frac{a}{3}\,x^3 \Big[_u^v + \frac{b}{2}\,x^2 \Big[_u^v + c\,x \Big[_u^v \right) \tag{6}$$

Using Equation (5) above and Appendix Equation (13) below, the solution to our parabola's second parameter (A_n) in Equation (4) above is...

$$A_n = \frac{1}{\pi} \int_u^v f(x)\,\cos(n\,x)\,\delta x = \frac{1}{\pi} \Big(a\,I(A)_3 + b\,I(A)_2 + c\,I(A)_1 \Big) \tag{7}$$

Note: See Appendix Equation (12) below for definitions of $I(A)_1$, $I(A)_2$ and $I(A)_3$ in Equation (6) above.

Using Equation (5) above and Appendix Equation (16) below, the solution to our parabola's third parameter (B_m) in Equation (4) above is...

$$B_m = \frac{1}{\pi} \int_0^{2\,\pi} f(x)\,\sin(m\,x)\,\delta x = \frac{1}{\pi} \Big(a\,I(B)_3 + b\,I(B)_2 + c\,I(B)_1 \Big) \tag{8}$$

Note: See Appendix Equation (15) below for definitions of $I(B)_1$, $I(B)_2$ and $I(B)_3$ in Equation (7) above.

The Solution To Our Hypothetical Problem

1. What is our equation for the Fourier Series approximated parabola?

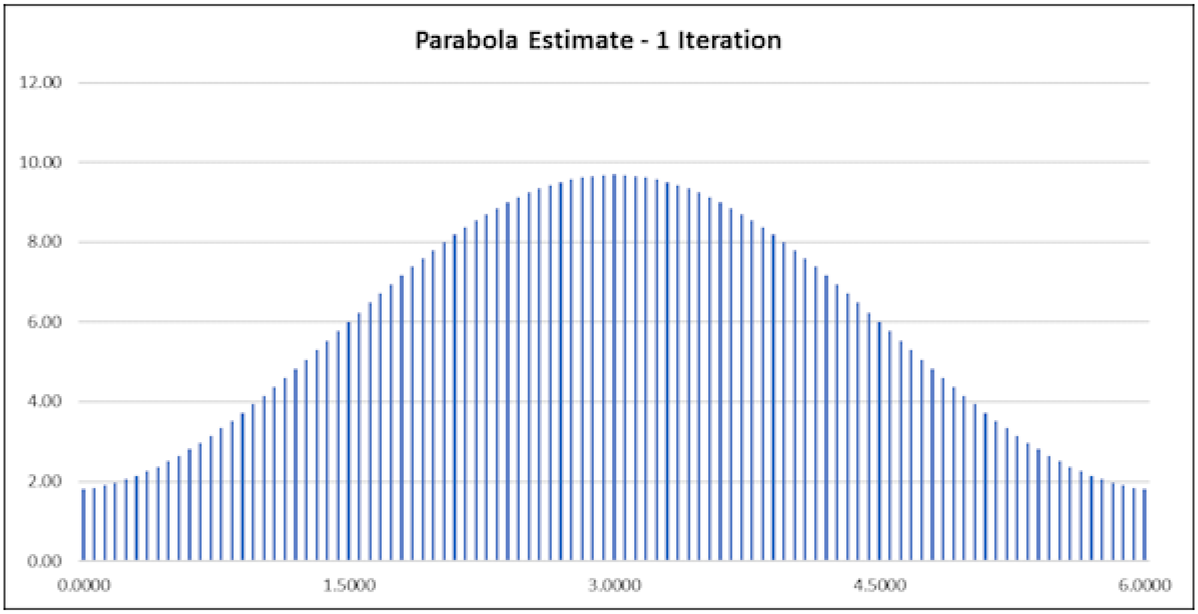
Using Equations (2), (6), (7) and (8) above, the equation to approximate our parabola is...

$$g(x) = A_0 + \sum_{n=1}^{\infty} A_n \cos(n\,x) + \sum_{m=1}^{\infty} B_m \sin(m\,x) \;\; \text{...where...} \;\; a = -1.0000,\; b = 6.0000,\; c = 0.0000 \tag{9}$$

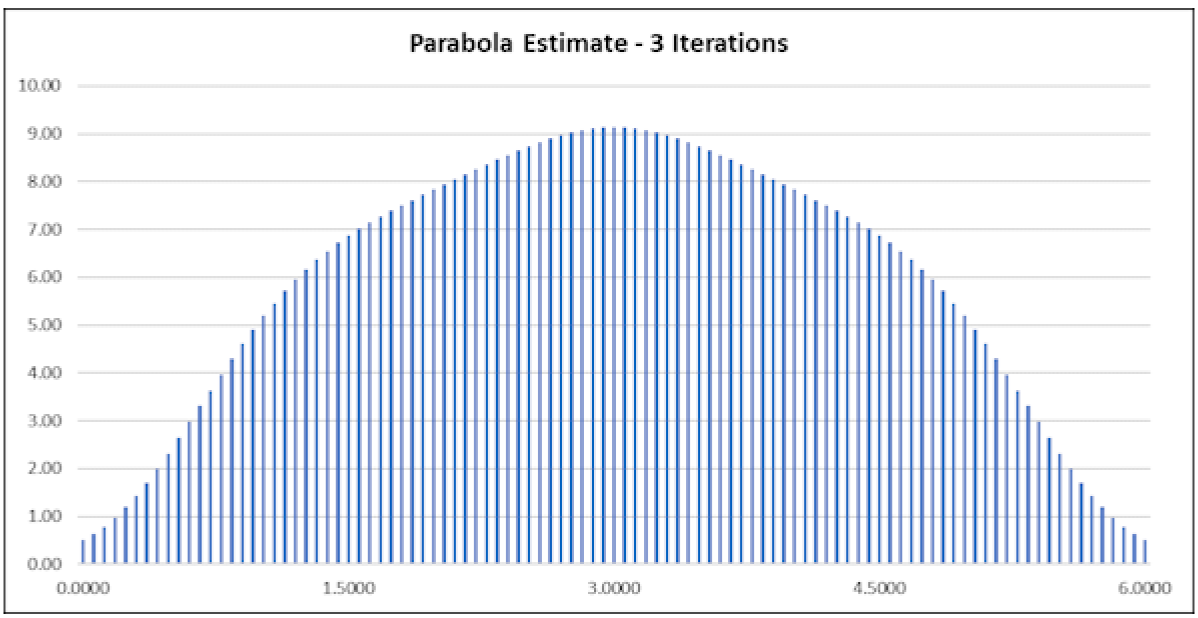
2. What are the values of the constant term and the first three coefficients for the sine and cosine terms?

Parameter	Description	Value	Reference
$A(0)$	constant term	5.7296	Equation (6)
$A(1)$	cosine term	-3.9215	Equation (7)
$A(2)$	cosine term	-0.9231	Equation (7)
$A(3)$	cosine term	-0.3700	Equation (7)
$B(1)$	sine term	0.5590	Equation (8)
$B(2)$	sine term	0.2686	Equation (8)
$B(3)$	sine term	0.1674	Equation (8)

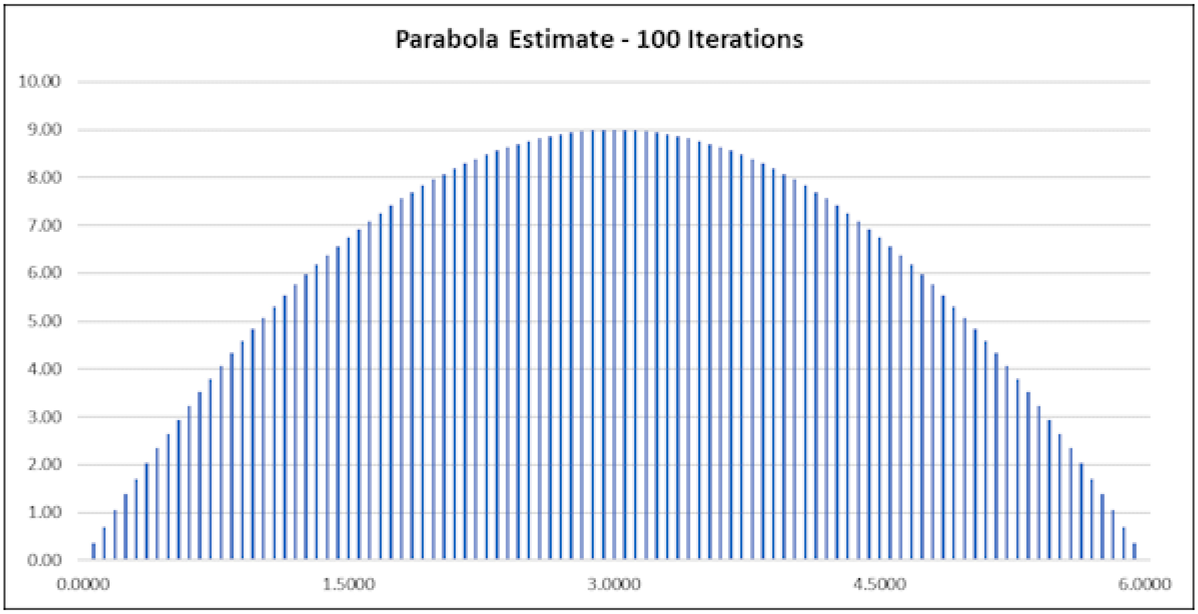
3. Graph the Fourier Series approximated parabola after 1 iteration.



4. Graph the Fourier Series approximated parabola after 3 iterations.



5. Graph the Fourier Series approximated parabola after 100 iterations.



Appendix

A. The solution to the following integral is...

$$
\begin{aligned}
A_0 &= \frac{1}{2\,\pi} \int_u^v \Big(a\,x^2 + b\,x + c \Big) \delta x \\
&= \frac{1}{2\,\pi} \Big(\int_u^v a\,x^2\,\delta x + \int_u^v b\,x\,\delta x + \int_u^v c\,\delta x \Big) \\
&= \frac{1}{2\,\pi} \Big(\frac{a}{3}\,x^3 \Big[_u^v + \frac{b}{2}\,x^2 \Big[_u^v + c\,x \Big[_u^v \Big)
\end{aligned}
\tag{10}
$$

B. The solution to the following integral is...

$$
\begin{aligned}
A_n &= \frac{1}{\pi} \int_u^v \Big(a\,x^2 + b\,x + c \Big) \cos(n\,x)\,\delta x \\
&= \frac{1}{\pi} \Big(a \int_u^v x^2\,\cos(n\,x)\,\delta x + b \int_u^v x\,\cos(n\,x)\,\delta x + c \int_u^v \cos(n\,x)\,\delta x \Big)
\end{aligned}
\tag{11}
$$

Note the solutions to the following integrals...

$$
\begin{aligned}
I(A)_1 &= \int_u^v \cos(n\,x)\,\delta x = \frac{1}{n}\,\sin(n\,x) \Big[_u^v \\
I(A)_2 &= \int_u^v x\,\cos(n\,x)\,\delta x = \frac{1}{n^2} \Big(n\,x\,\sin(n\,x) - \cos(n\,x) \Big) \Big[_u^v \\
I(A)_3 &= \int_u^v x^2\,\cos(n\,x)\,\delta x = \frac{1}{n} \Big(x^2\,\sin(n\,x) - \frac{2}{n^2} \Big[\sin(n\,x) - n\,x\,\cos(n\,x) \Big] \Big) \Big[_u^v
\end{aligned}
\tag{12}
$$

Using the integral solutions in Equation (12) above, we can rewrite Equation (11) above as...

$$
A_n = \frac{1}{\pi} \Big(a\,I(A)_3 + b\,I(A)_2 + c\,I(A)_1 \Big) \tag{13}
$$

C. The solution to the following integral is...

$$B_m = \frac{1}{\pi} \int\limits_u^v \bigg(a\,x^2 + b\,x + c\bigg) \sin(m\,x)\,\delta x$$
$$= \frac{1}{\pi} \Bigg(a \int\limits_u^v x^2\,\sin(m\,x)\,\delta x + b \int\limits_u^v x\,\sin(m\,x)\,\delta x + c \int\limits_u^v \sin(m\,x)\,\delta x\Bigg) \tag{14}$$

Note the solutions to the following integrals...

$$I(B)_1 = \int\limits_u^v \sin(m\,x)\,\delta x = -\frac{1}{m}\,\cos(m\,x)\Bigg[_u^v$$
$$I(B)_2 = \int\limits_u^v x\,\sin(m\,x)\,\delta x = \frac{1}{m^2}\Bigg(\sin(m\,x) - m\,x\,\cos(m\,x)\Bigg)\Bigg[_u^v$$
$$I(B)_3 = \int\limits_u^v x^2\,\sin(m\,x)\,\delta x = \frac{1}{m}\Bigg(\frac{2}{m^2}\bigg[m\,x\,\sin(m\,x) + \cos(m\,x)\bigg] - x^2\,\cos(m\,x)\Bigg)\Bigg[_u^v \tag{15}$$

Using the integral solutions in Equation (15) above, we can rewrite Equation (14) above as...

$$B_m = \frac{1}{\pi}\Bigg(a\,I(B)_3 + b\,I(B)_2 + c\,I(B)_1\Bigg) \tag{16}$$

References

[1] Gary Schurman, *The Fourier Series - Derivatives and Integrals Of Trigonometric Functions,* September, 2023.

[2] Gary Schurman, *The Fouier Series - The Fourier Expansion,* September, 2023.